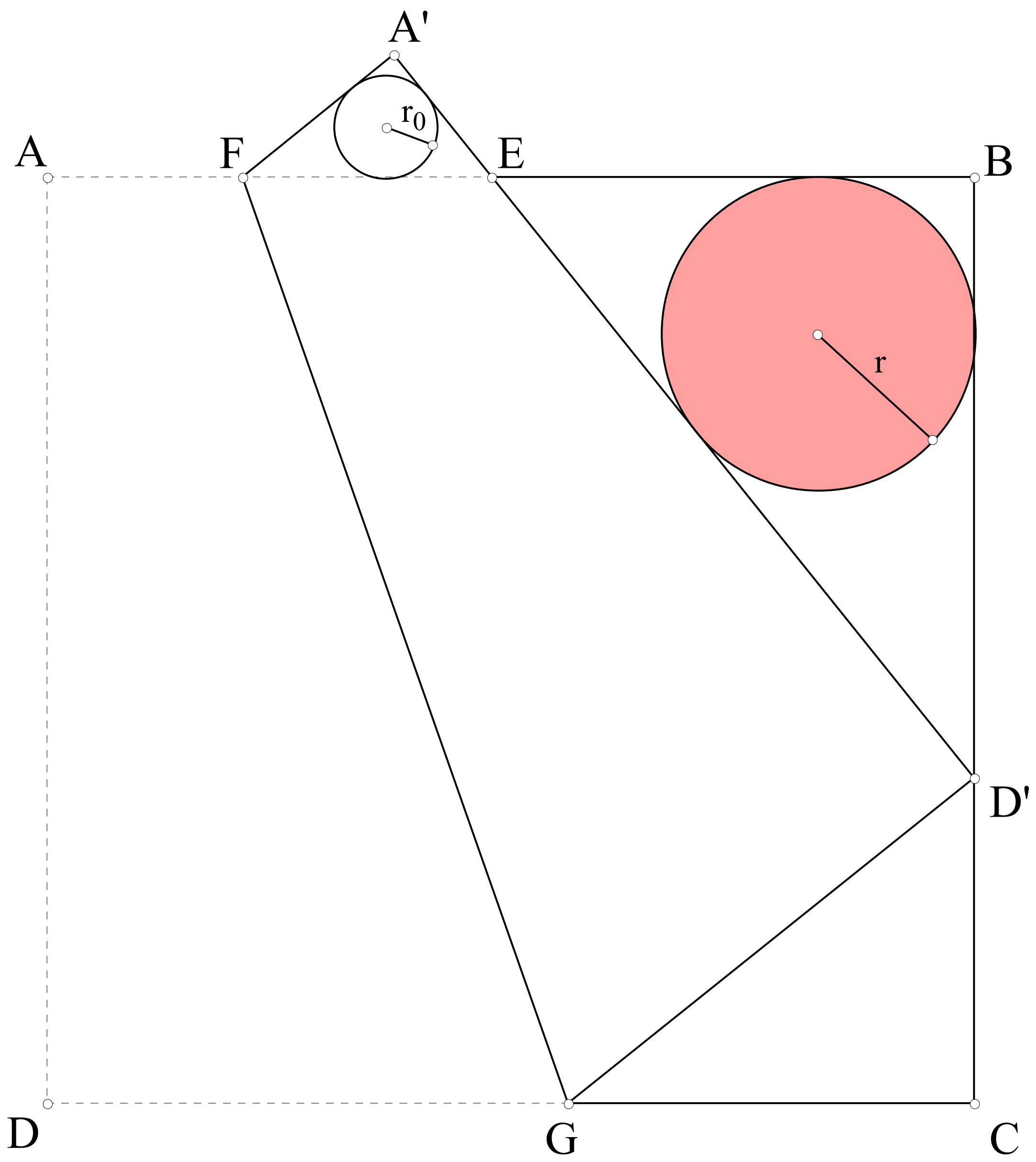
Japanese Temple Geometry Problem J3.1



Triangle $A'FE$ is right-angled and $AF = FA'$ (1.1)

Let r be the Incircle of the triangle BDE and r_0 be the Incircle of the triangle $A'EF$.

Then $$ED' = A'D' - A'E \tag{1.2}$$

and $$BE = AB - (AF + FE)$$

Therefore by (1.1) $$BE = AB - (FA' + FE) \tag{1.3}$$

From Appendix 1 we have that

$$2r_0 = A'F + A'E - FE \tag{1.4}$$

Therefore, from similar triangles $A'FE$ and BED' we have

That is

$$r : BE = r_0 : A'E$$
$$r \cdot A'E = r_0 \cdot BE$$
$$r \cdot A'E = r_0 \cdot \left(AB - \left(FA' + FE \right) \right) \quad \text{by (1.3)}$$

Hence $$r \cdot A'E = r_0 AB - r_0 \left(FA' + FE \right) \tag{1.5}$$

Similarly, we have $$r : D'E = r_0 : EF$$

That is $$r \cdot EF = r_0 \cdot D'E$$
$$r \cdot EF = r_0 \cdot \left(A'D' - A'E \right) \quad \text{by (1.2)}$$

Hence $$r \cdot EF = r_0 A'D' - r_0 A'E \tag{1.6}$$

(1.6) - (1.5) gives $$r\left(EF - A'E \right) = r_0 \left[A'D' - A'E - AB + FA' + FE \right]$$

But $A'D' = AB$ (Sides of the square $ABCD$)

Hence $$r\left(EF - A'E \right) = r_0 \left(FA' + FE - A'E \right) \tag{1.7}$$

$$r\left(EF - A'E \right) = \tfrac{1}{2}\left(A'F + A'E - FE \right)\left(FA' + FE - A'E \right) \quad \text{by (1.4)}$$

Therefore,

$$r\left(EF - A'E \right) = \tfrac{1}{2}\left(A'F + \left(A'E - FE \right) \right)\left(A'F - \left(A'E - FE \right) \right)$$
$$= \tfrac{1}{2}\left[\left(A'F \right)^2 - \left(A'E - FE \right)^2 \right]$$
$$= \tfrac{1}{2}\left[\left(A'F \right)^2 - \left(A'E \right)^2 - \left(FE \right)^2 + 2\left(A'E \right)\left(FE \right) \right]$$

Hence, by Pythagoras in triangle $A'FE$ we have

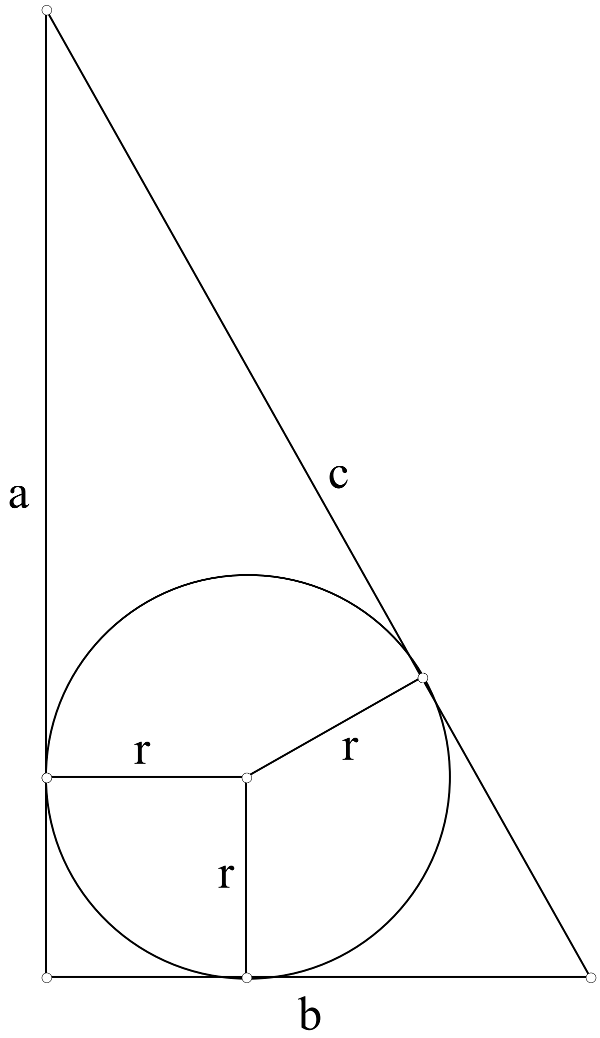
$$r\left(EF - A'E \right) = \tfrac{1}{2}\left[-\left(A'E \right)^2 - \left(A'E \right)^2 + 2\left(A'E \right)\left(FE \right) \right]$$

Therefore,

$$r\left(EF - A'E \right) = A'E\left(FE - A'E \right) \tag{1.8}$$

But $EF \neq A'E$ when $D' \to B \quad \Rightarrow \qquad r = A'E \quad \square$

Appendix 1



It is easily seen that

$$c = (a - r) + (b - r)$$

Hence

$$\boxed{2r = a + b - c}$$