

## Inversion Theory

(1)

Given a *Circle of Inversion*,  $T(k)$ , where the point  $T$  is called the *Centre of Inversion* and  $k$  is called the *Radius (or Constant) of Inversion*, then the following results are determined:-

2. If  $P$  is any given point, we can find a point  $P'$  on the line through  $TP$  such that

$$TP \times TP' = k^2$$

$P'$  is known as the *Inverse of P*.

3. If  $P$  and  $Q$  are any two points with inverses  $P'$  and  $Q'$  then

$$|P'Q'| = \frac{k^2}{|(TP)(TQ)|} |PQ| \quad \text{and} \quad |PQ| = \frac{k^2}{|(TP')(TQ')|} |P'Q'|$$

4. As an extension to (1) above it can be shown that:

$$x' = \frac{k^2 x}{x^2 + y^2} \quad \text{and} \quad y' = \frac{k^2 y}{x^2 + y^2}$$

and also from (1) above  $\sqrt{x^2 + y^2} \times \sqrt{x'^2 + y'^2} = k^2$ .

Taken together  $\Rightarrow \quad x = \frac{k^2 x'}{x'^2 + y'^2} \quad \text{and} \quad y = \frac{k^2 y'}{x'^2 + y'^2}$ . (See (2) above)

Where  $x, x', y$  and  $y'$  are distances from the Centre of Inversion to the points  $P(x, y)$  and  $P'(x', y')$ , respectively.

5. The inverse of a Line (Infinite Line) that does NOT pass through Centre of Inversion,  $T$ , is an Arc of a Circle (Full Circle) that passes through the  $T$ , otherwise if the Line passes through  $T$  then it is inverted to a Line, itself through  $T$ , and coincident to the Object Line.
6. The inverse of an Arc of a Circle (Full Circle) that does NOT pass through Centre of Inversion,  $T$ , is an Arc of a Circle (Full Circle) NOT through  $T$ , otherwise it is a line that also does NOT pass through  $T$ .

Of **particular** use is the following circle result:

7. If any Circle  $O(r)$  inverts into the Circle  $O'(r')$  then

$$\frac{r}{r'} = \frac{k^2}{(t')^2},$$

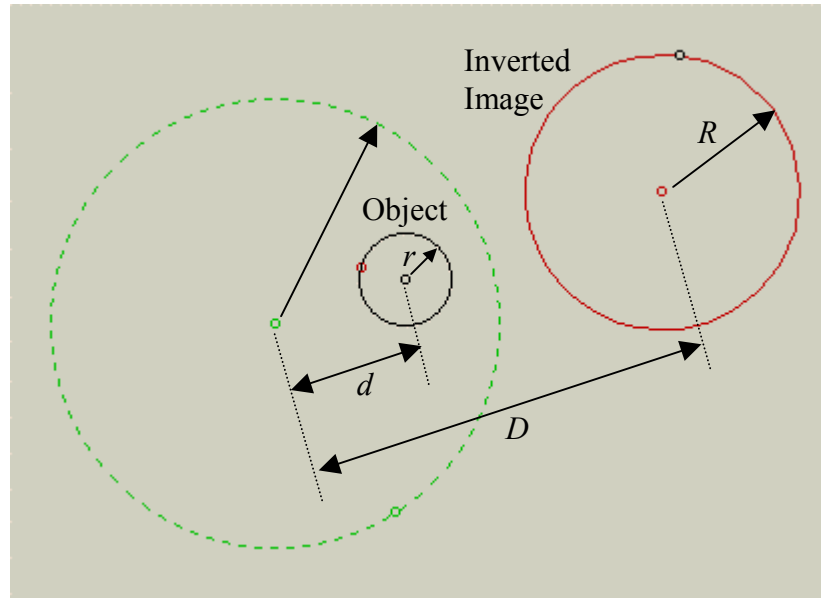
where  $t'$  is the length of a Tangent from  $T$  to the Inverse Circle  $O'(r')$ .

## Inversion Theory – Continued...

(2)

An alternative notation that is particularly useful when inverting circles is explained below

1. If  $R$  is the radius of the Inverted Image and  $r$  is the radius of the Original circle, then the following results hold:



To find the object radius,  $r$ , we use  $r = \frac{Rk^2}{|D^2 - R^2|}$ , which can be seen to be equivalent to the equation in rule (6) above (using Pythagoras' Theorem on denominator), but may be easier to use and recall.

To find the distance,  $d$ , we use

$$d = \frac{Dk^2}{|D^2 - R^2|}.$$

As is often the case, we will need to find the centre coordinates of the object circles,  $(x, y)$ .

We do this by using the two formulae:

$$x = \frac{Xk^2}{|D^2 - R^2|} \quad \text{and} \quad y = \frac{Yk^2}{|D^2 - R^2|},$$

where  $(X, Y)$  are the coordinates of the centre of the inverted image circle.

Clearly, if the radius of the Inverted Image Circle and coordinates of the centre of the Inverted Image Circle are difficult to find from the diagram, then it will also be very difficult to find the *actual* radius and centre coordinates of the Object Circle. Therefore, the success of the method of Inversion relies heavily on the skill of the user in choosing the position and size of the *Circle of Inversion* so as to make the Geometry of the entire Inverted Image as simple as possible.

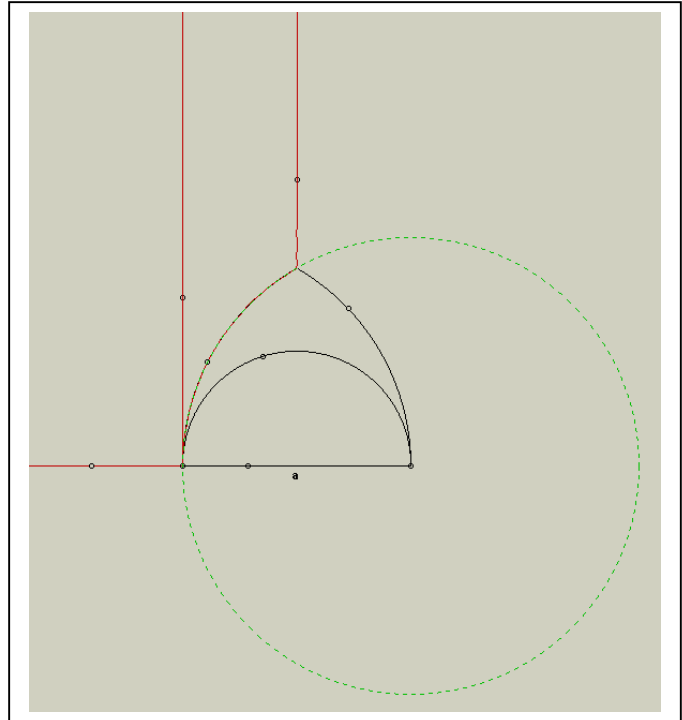
### Inversion Exercise (3):- Continued...

(3)

In this problem our aim is to use Inversion Theory to find the radii and centres of the family of circles sandwiched between the outer Arch and the inner Semicircle shown below. Also shown below, after some trial and error, is a position and size of the Circle of Inversion that gives us a relatively simple geometry of the Inverted Image (though there may be other equally valid positions and sizes).

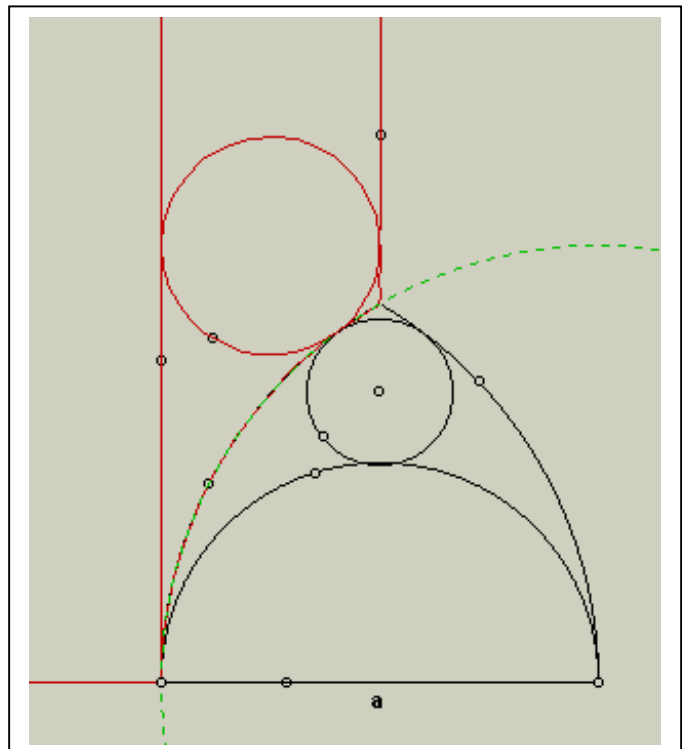
We have chosen the Centre of the Circle of Inversion as the bottom right point of the outer Arch and the Radius of the Circle of Inversion as  $a$ , the width of the Arch (or the diameter of the internal Semicircle).

The Inverted Images of the outer Arches, the Semicircle and the Base of the Arch are shown in red. From the SketchPad file, drag the random points that lie on either the Arch or its Inverted Image to show exactly how the inversion process has worked on each separate object.



Shown here is how the actual Top, Tangent Circle in the Arch will be Inverted.

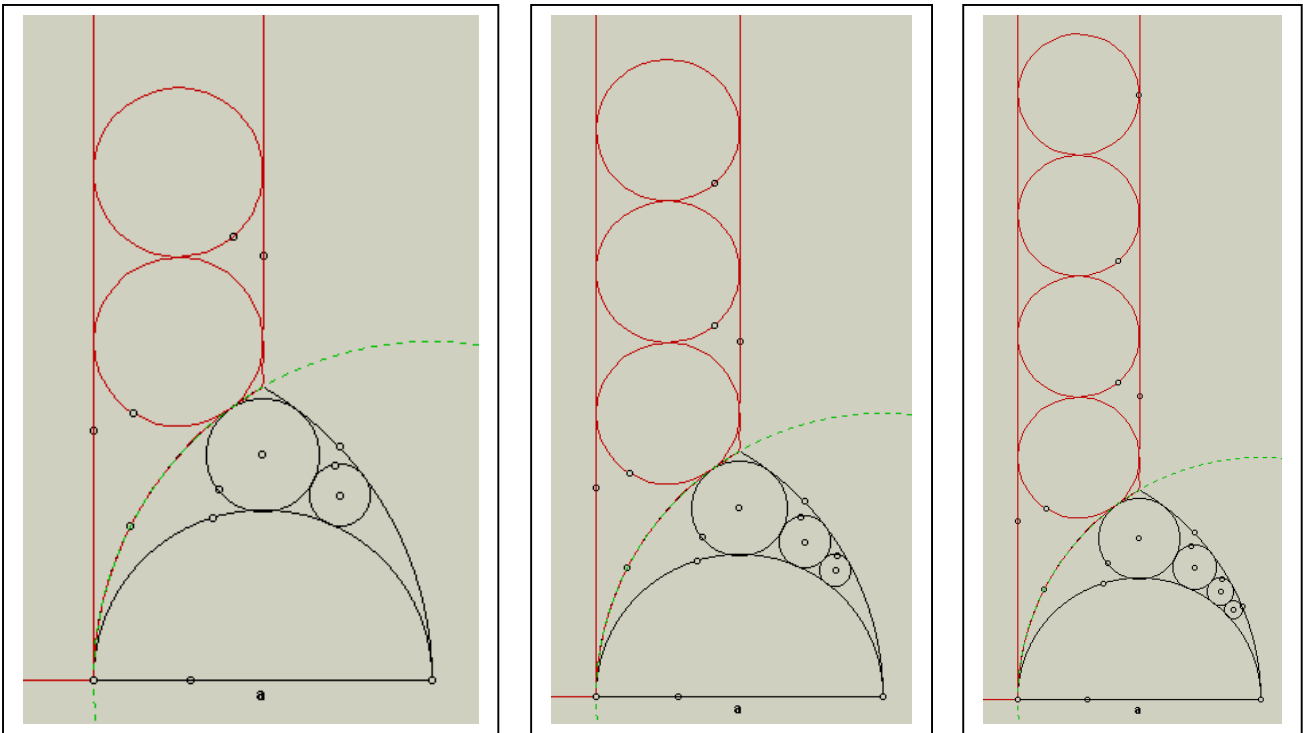
Over the page we see further 'Side' Tangent Circles.



### Inversion Exercise (3):- Continued...

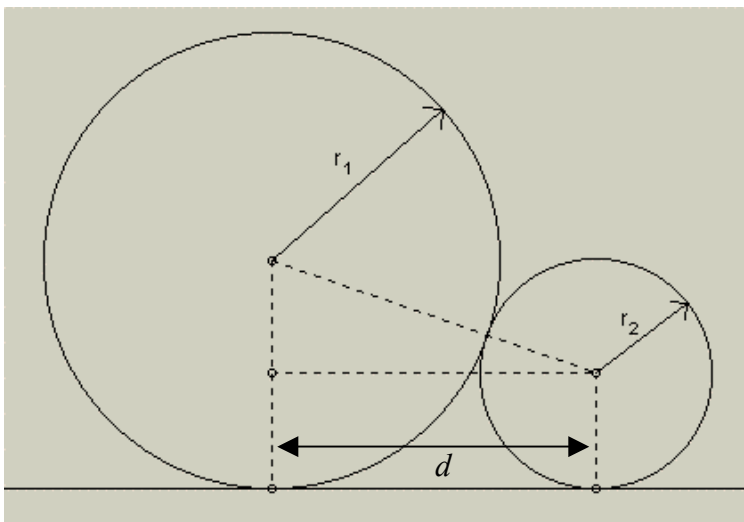
(4)

The sequence the first four internal Tangent Circles and their Inverted Images are shown below.



Before we continue a result is needed:-

Rule (8) *Distance between Centres of Two Circles tangential to each other and a common line.*



By Pythagoras we see that:

$$(r_1 + r_2)^2 = (r_1 - r_2)^2 + d^2,$$

where  $d$  is the horizontal distance between centres.

Therefore,

$$d^2 = (r_1 + r_2)^2 - (r_1 - r_2)^2$$

$$= (\overline{r_1 + r_2 + r_1 - r_2})(\overline{r_1 + r_2 - r_1 - r_2})$$

$$= (2r_1)(2r_2)$$

$$= 4r_1r_2, \text{ giving}$$

$$d = 2\sqrt{r_1r_2}$$

### Inversion Exercise (3):- Continued...

(5)

Consider just the Inverted Image and ignore, for clarity, the Original Arch structure.

Here then we see that each of the Inverted Circles has a radius of  $\frac{1}{4}a$ . In order that we may apply either of Rule 6 or 7 from the above Theory Box to determine both the radii and centres of the *Original Tangent Circles*, we must first find either  $t'_0$  or  $D_0$ , respectively.

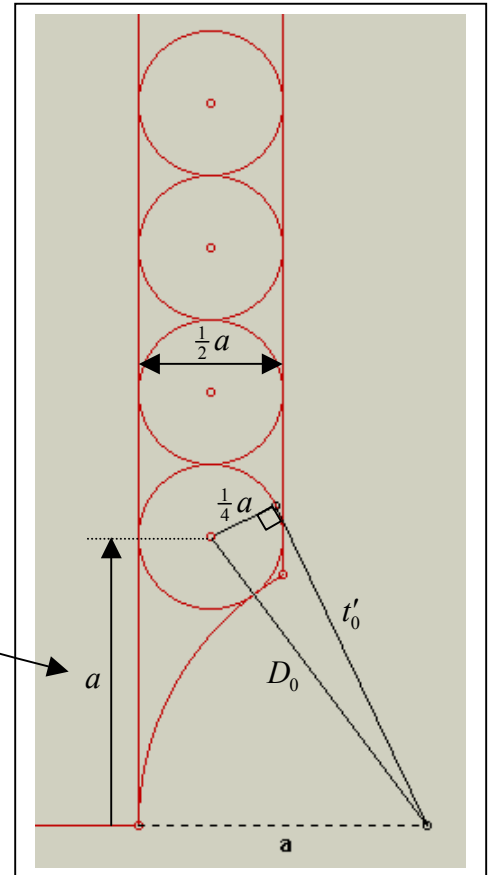
Using the above Rule 8 with the bottom partial Arch and first Inverted Circle with radii  $r_1 = \frac{1}{4}a$  and  $r_2 = a$ , gives:

$$\begin{aligned} \text{Distance between Centres} &= 2\sqrt{r_1 r_2} \\ &= 2\sqrt{\left(\frac{1}{4}a\right)(a)} \\ &= a \end{aligned}$$

Therefore, we have

$$\begin{aligned} D_0^2 &= \left(\frac{3}{4}a\right)^2 + (a)^2 \\ k &= a \\ R_0 &= \frac{1}{4}a \end{aligned}$$

$$r_0 = \frac{R_0 k^2}{|D_0^2 - R_0^2|} = \frac{\left(\frac{1}{4}a\right)(a)^2}{\left|\left(\frac{3}{4}a\right)^2 + (a)^2 - \left(\frac{1}{4}a\right)^2\right|} = \frac{1}{6}a$$



Therefore, the radius of the Top Tangent Circle is  $r_0 = \frac{1}{6}a$ .

Similarly, for the coordinates of the centre of the Top Tangent Circle, relative to the Centre of the Circle of Inversion are given by:-

$$x_0 = \frac{X_0 k^2}{|D_0^2 - R_0^2|} = \frac{\left(\frac{3}{4}a\right)(a)^2}{\left|\left(\frac{3}{4}a\right)^2 + (a)^2 - \left(\frac{1}{4}a\right)^2\right|} = \frac{1}{2}a$$

and

$$y_0 = \frac{Y_0 k^2}{|D_0^2 - R_0^2|} = \frac{(a)(a)^2}{\left|\left(\frac{3}{4}a\right)^2 + (a)^2 - \left(\frac{1}{4}a\right)^2\right|} = \frac{2}{3}a.$$

Therefore, the coordinates of the centre of the Top Tangent Circle is given by:

$$(x_0, y_0) = \left(\frac{1}{2}a, \frac{2}{3}a\right)$$

**NB.** It seems clear that the easiest method will be to use the Theory from Rule 7. above in the Theorem Box on page 2, rather than Rule 6.

### Inversion Exercise (3):- Continued...

(6)

In this case we have:

$$D_1^2 = \left(\frac{3}{4}a\right)^2 + \left(a + \frac{1}{2}a\right)^2$$

$$k = a$$

$$R_1 = \frac{1}{4}a$$

$$r_1 = \frac{R_1 k^2}{|D_1^2 - R_1^2|} = \frac{\left(\frac{1}{4}a\right)(a)^2}{\left|\left(\frac{3}{4}a\right)^2 + \left(a + \frac{1}{2}a\right)^2 - \left(\frac{1}{4}a\right)^2\right|} = \frac{1}{11}a.$$

Therefore, the radius of the Top Tangent Circle is  $r_1 = \frac{1}{11}a$ .

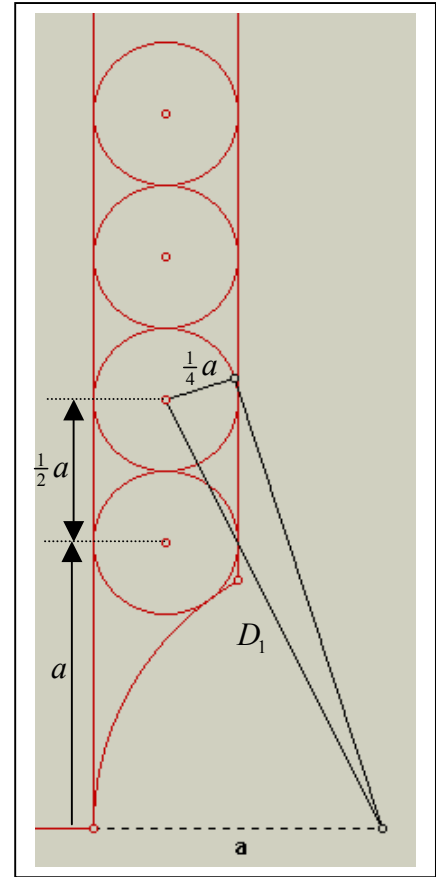
Also, the centre coordinates are found from:

$$x_1 = \frac{X_1 k^2}{|D_1^2 - R_1^2|} = \frac{\left(\frac{3}{4}a\right)(a)^2}{\left|\left(\frac{3}{4}a\right)^2 + \left(a + \frac{1}{2}a\right)^2 - \left(\frac{1}{4}a\right)^2\right|} = \frac{3}{11}a,$$

$$y_1 = \frac{Y_1 k^2}{|D_1^2 - R_1^2|} = \frac{\left(a + \frac{1}{2}a\right)(a)^2}{\left|\left(\frac{3}{4}a\right)^2 + \left(a + \frac{1}{2}a\right)^2 - \left(\frac{1}{4}a\right)^2\right|} = \frac{6}{11}a.$$

Therefore, the coordinates of the centre of the Top Tangent Circle is given by:

$$(x_1, y_1) = \left(\frac{3}{11}a, \frac{6}{11}a\right)$$



In this case we have:

$$D_2^2 = \left(\frac{3}{4}a\right)^2 + \left(a + \frac{1}{2}a + \frac{1}{2}a\right)^2 = \left(\frac{3}{4}a\right)^2 + \left(a + 2 \times \frac{1}{2}a\right)^2$$

$$k = a$$

$$R_2 = \frac{1}{4}a$$

$$r_2 = \frac{R_2 k^2}{|D_2^2 - R_2^2|} = \frac{\left(\frac{1}{4}a\right)(a)^2}{\left|\left(\frac{3}{4}a\right)^2 + \left(a + 2 \times \frac{1}{2}a\right)^2 - \left(\frac{1}{4}a\right)^2\right|} = \frac{1}{18}a.$$

Therefore, the radius of the Top Tangent Circle is  $r_2 = \frac{1}{18}a$ .

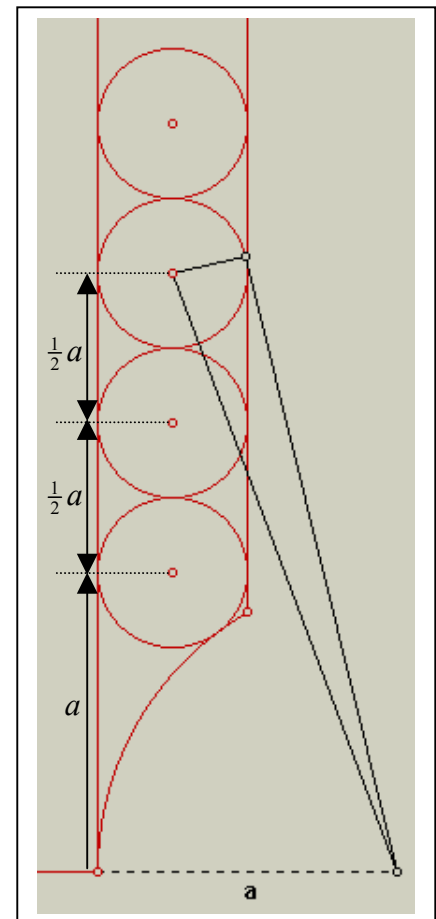
Also, the centre coordinates are found from:

$$x_2 = \frac{X_2 k^2}{|D_2^2 - R_2^2|} = \frac{\left(\frac{3}{4}a\right)(a)^2}{\left|\left(\frac{3}{4}a\right)^2 + \left(a + 2 \times \frac{1}{2}a\right)^2 - \left(\frac{1}{4}a\right)^2\right|} = \frac{1}{6}a,$$

$$y_2 = \frac{Y_2 k^2}{|D_2^2 - R_2^2|} = \frac{\left(a + 2 \times \frac{1}{2}a\right)(a)^2}{\left|\left(\frac{3}{4}a\right)^2 + \left(a + 2 \times \frac{1}{2}a\right)^2 - \left(\frac{1}{4}a\right)^2\right|} = \frac{4}{9}a.$$

Therefore, the coordinates of the centre of the Top Tangent Circle is given by:

$$(x_2, y_2) = \left(\frac{1}{6}a, \frac{4}{9}a\right)$$



### Inversion Exercise (3):- Continued...

(7)

In this case we have:

$$D_3^2 = \left(\frac{3}{4}a\right)^2 + \left(a + \frac{1}{2}a + \frac{1}{2}a + \frac{1}{2}a\right)^2 = \left(\frac{3}{4}a\right)^2 + \left(a + 3 \times \frac{1}{2}a\right)^2$$

$$k = a$$

$$R_3 = \frac{1}{4}a$$

$$r_3 = \frac{R_3 k^2}{|D_3^2 - R_3^2|} = \frac{\left(\frac{1}{4}a\right)(a)^2}{\left|\left(\frac{3}{4}a\right)^2 + \left(a + 3 \times \frac{1}{2}a\right)^2 - \left(\frac{1}{4}a\right)^2\right|} = \frac{1}{27}a.$$

Therefore, the radius of the Top Tangent Circle is  $r_3 = \frac{1}{27}a$ .

Also, the centre coordinates are found from:

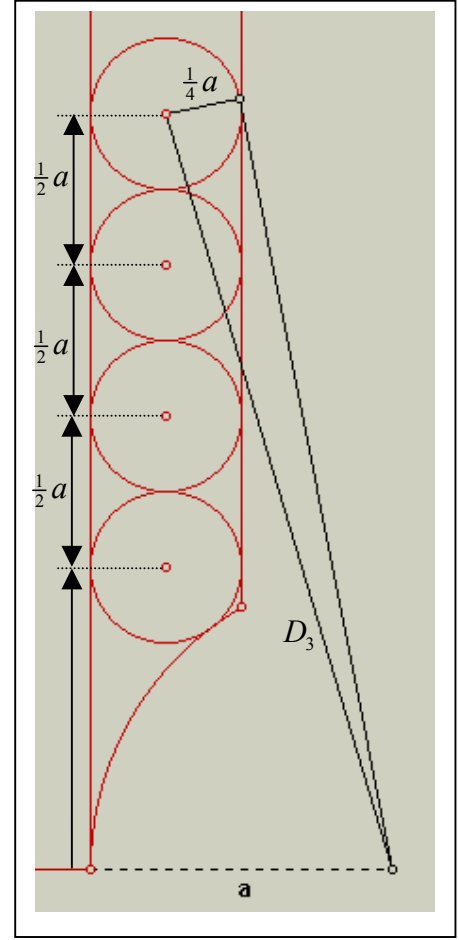
$$x_3 = \frac{X_3 k^2}{|D_3^2 - R_3^2|} = \frac{\left(\frac{3}{4}a\right)(a)^2}{\left|\left(\frac{3}{4}a\right)^2 + \left(a + 3 \times \frac{1}{2}a\right)^2 - \left(\frac{1}{4}a\right)^2\right|} = \frac{1}{9}a,$$

$$y_3 = \frac{Y_3 k^2}{|D_3^2 - R_3^2|} = \frac{\left(a + 3 \times \frac{1}{2}a\right)(a)^2}{\left|\left(\frac{3}{4}a\right)^2 + \left(a + 3 \times \frac{1}{2}a\right)^2 - \left(\frac{1}{4}a\right)^2\right|} = \frac{10}{27}a.$$

Therefore, the coordinates of the centre of the Top Tangent Circle is given by:

$$(x_3, y_3) = \left(\frac{1}{9}a, \frac{10}{27}a\right)$$

In general then, for  $n$  Internal, Tangent Circles and their associated Inverted Image Circles, we observe the following patterns:



The radius of the  $n^{\text{th}}$  Internal, Tangent Circle,  $r_n$ , is given by:

$$r_n = \frac{R_n k^2}{|D_n^2 - R_n^2|} = \frac{\left(\frac{1}{4}a\right)(a)^2}{\left|\left(\frac{3}{4}a\right)^2 + \left(a + n \times \frac{1}{2}a\right)^2 - \left(\frac{1}{4}a\right)^2\right|} = \frac{a}{2 + (n+2)^2},$$

and the location of the Centre Coordinates, with respect to the Centre of the Circle of Inversion, is given by:

$$x_n = \frac{X_n k^2}{|D_n^2 - R_n^2|} = \frac{\left(\frac{3}{4}a\right)(a)^2}{\left|\left(\frac{3}{4}a\right)^2 + \left(a + n \times \frac{1}{2}a\right)^2 - \left(\frac{1}{4}a\right)^2\right|} = \frac{3a}{2 + (n+2)^2} \equiv 3r_n$$

and

$$y_n = \frac{Y_n k^2}{|D_n^2 - R_n^2|} = \frac{\left(a + n \times \frac{1}{2}a\right)(a)^2}{\left|\left(\frac{3}{4}a\right)^2 + \left(a + n \times \frac{1}{2}a\right)^2 - \left(\frac{1}{4}a\right)^2\right|} = \frac{2(n+2)a}{2 + (n+2)^2} \equiv 2(n+2)r_n$$

That is:

$$\boxed{r_n = \frac{a}{2 + (n+2)^2}, \quad x_n = \frac{3a}{2 + (n+2)^2} \quad \text{and} \quad y_n = \frac{2(n+2)a}{2 + (n+2)^2}}$$

which are each valid for  $n \in \mathbb{N}_0$ , the case  $n = 0$  giving the Top Tangent Circle.