Inversion Theory

Given a *Circle of Inversion*, *T(k)*, where the point *T* is called the *Centre of Inversion* and *k* is called the *Radius (or Constant) of Inversion*, then the following results are determined:-

2. If *P* is any given point, we can find a point P' on the line through *TP* such that

$$TP \times TP' = k^2$$

   P' is known as the *Inverse of P*.

3. If *P* and *Q* are any two points with inverses P' and Q' then

$$|P'Q'| = \frac{k^2}{|(TP)(TQ)|}|PQ| \quad \text{and} \quad |PQ| = \frac{k^2}{|(TP')(TQ')|}|P'Q'|$$

4. As an extension to (1) above it can be shown that:

$$x' = \frac{k^2 x}{x^2 + y^2} \quad \text{and} \quad y' = \frac{k^2 y}{x^2 + y^2}$$

   and also from (1) above $\sqrt{x^2 + y^2} \times \sqrt{x'^2 + y'^2} = k^2$.

   Taken together $\Rightarrow$ $x = \dfrac{k^2 x'}{x'^2 + y'^2}$ and $y = \dfrac{k^2 y'}{x'^2 + y'^2}$. (See (2) above)

   Where x, x', y and y' are distances from the Centre of Inversion to the points $P(x, y)$ and $P'(x', y')$, respectively.

5. The inverse of a Line (Infinite Line) that does NOT pass through Centre of Inversion, *T*, is an Arc of a Circle (Full Circle) that passes through the *T*, otherwise if the Line passes through *T* then it is inverted to a Line, itself through *T*, and coincident to the Object Line.

6. The inverse of an Arc of a Circle (Full Circle) that does NOT pass through Centre of Inversion, *T*, is an Arc of a Circle (Full Circle) NOT through *T*, otherwise it is a line that also does NOT pass through *T*.

   Of ***particular*** use is the following circle result:

7. If any Circle $O(r)$ inverts into the Circle $O'(r')$ then

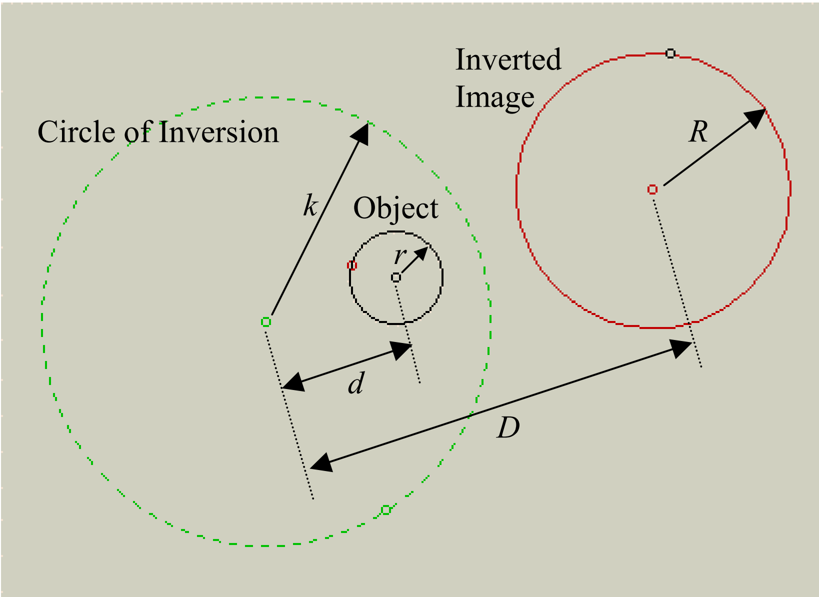
$$\frac{r}{r'} = \frac{k^2}{(t')^2},$$

   where t' is the length of a Tangent from *T* to the Inverse Circle $O'(r')$.

Inversion Theory – Continued…

An alternative notation that is particularly useful when inverting circles is explained below

1. If R is the radius of the Inverted Image and r is the radius of the Original circle, then the following results hold:



To find the object radius, r, we use $r = \frac{Rk^2}{|D^2 - R^2|}$, which can be seen to be equivalent to the equation in rule (6) above (using Pythagoras' Theorem on denominator), but may be easier to use and recall.

To find the distance, d, we use

$$d = \frac{Dk^2}{|D^2 - R^2|}.$$

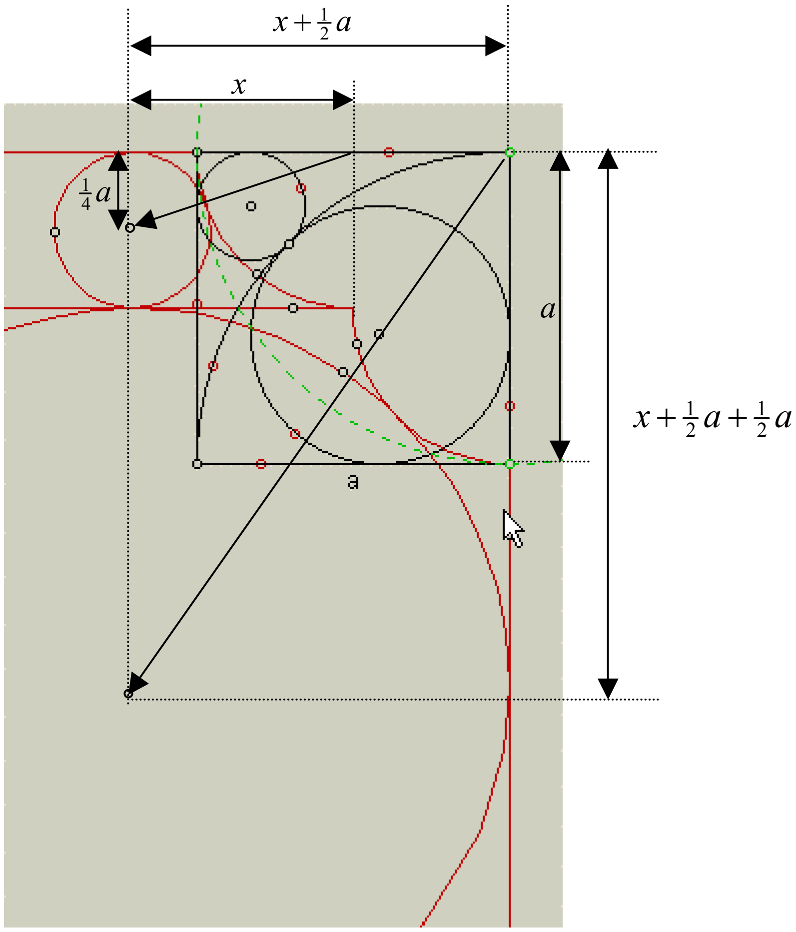
As is often the case, we will need to find the centre coordinates of the object circles, (x, y). We do this by using the two formulae:

$$x = \frac{Xk^2}{|D^2 - R^2|} \text{ and } y = \frac{Yk^2}{|D^2 - R^2|},$$

where (X, Y) are the coordinates of the centre of the inverted image circle.

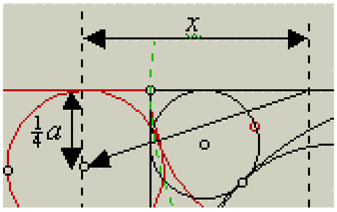
Clearly, if the radius of the Inverted Image Circle and coordinates of the centre of the Inverted Image Circle are difficult to find from the diagram, then it will also be very difficult to find the *actual* radius and centre coordinates of the Object Circle. Therefore, the success of the method of Inversion relies heavily on the skill of the user in choosing the position and size of the *Circle of Inversion* so as to make the Geometry of the entire Inverted Image as simple as possible.

Inversion Exercise:- To find the Radii of the Internal Circles



In the diagram the red lines are the inversions of the original black lined shape in which we need to find the radii of the internal circles to the square of side length a.

Consider the right-angled triangle taken from this picture:-



The hypotenuse can be seen to be $\frac{1}{2}a+\frac{1}{4}a=\frac{3}{4}a$. Therefore, x can be found to be $\frac{\sqrt{2}}{2}a$. Similarly, the radius of the larger inverted circle is seen as $\frac{1}{2}a\left(1+\sqrt{2}\right)$

Now, using Rule 5 from the Theory Box above we have the result:

$$\frac{r}{r'}=\frac{k^2}{(t')^2} \qquad \text{--(1)}$$

So for the smaller Circle:

r is to be found ,

$r'=\frac{1}{4}a$,

$k=a$ (radius of the Circle of Inversion)

$t'=x+\frac{1}{2}a=\frac{\sqrt{2}}{2}a+\frac{1}{2}a=\frac{1}{2}a\left(1+\sqrt{2}\right)$

Putting these into Eq.(1) quickly yields the result:-

$$r_{\text{Small}}=\left(3-2\sqrt{2}\right)a$$

Similarly, for the larger Circle:

r is to be found ,

$r'=\frac{1}{2}a\left(1+\sqrt{2}\right)$,

$k=a$ (radius of the Circle of Inversion)

$t'=\frac{1}{2}a\left(1+\sqrt{2}\right)+\frac{1}{2}a=\frac{1}{2}a\left(2+\sqrt{2}\right)$

Putting these into Eq.(1) quickly yields the result:-

$$r_{\text{Large}}=\left(\sqrt{2}-1\right)a$$

Note that, due to the symmetry of the original shape, it is not entirely necessarily to calculate coordinates of centres of the circles using Rules (2), (3) or (7) above. In fact it is *much* easier to use the original shape together with the new-found radii in order to calculate the respective centre coordinates.

However, to show how it is done we use the formulae from Rule 7. in the above Theory Box.

The centre of the Circle of Inversion is at the top right of the square, therefore we measure x as positive to the left and y positive down, without loss of generality.

For the small circle we have

$$\begin{aligned} D^2 - R^2 &= \left(t'\right)^2 \\ &= \left(\tfrac{1}{2}a\left(\sqrt{2}+1\right)\right)^2 \\ &= \tfrac{1}{4}a^2\left(3+2\sqrt{2}\right) \end{aligned}$$

Therefore, $$x = \frac{X k^2}{\left|D^2 - R^2\right|} = \frac{\tfrac{1}{2}a\left(\sqrt{2}+1\right)a^2}{\left(\tfrac{1}{2}a\left(\sqrt{2}+1\right)\right)^2} = 2a\left(\sqrt{2}-1\right)$$

and $$y = \frac{Y k^2}{\left|D^2 - R^2\right|} = \frac{\left(\tfrac{1}{4}a\right)a^2}{\left(\tfrac{1}{2}a\left(\sqrt{2}+1\right)\right)^2} = a\left(3-2\sqrt{2}\right)$$

Similarly, for the large circle we have

$$\begin{aligned} D^2 - R^2 &= \left(t'\right)^2 \\ &= \left(\tfrac{1}{2}a\left(2+\sqrt{2}\right)\right)^2 \\ &= \tfrac{1}{2}a^2\left(3+2\sqrt{2}\right) \end{aligned}$$

Therefore, $$x = \frac{X k^2}{\left|D^2 - R^2\right|} = \frac{\tfrac{1}{2}a\left(\sqrt{2}+1\right)a^2}{\left(\tfrac{1}{2}a\left(2+\sqrt{2}\right)\right)^2} = a\left(\sqrt{2}-1\right)$$

and $$y = \frac{Y k^2}{\left|D^2 - R^2\right|} = \frac{\left(\tfrac{1}{2}a\left(2+\sqrt{2}\right)\right)a^2}{\left(\tfrac{1}{2}a\left(2+\sqrt{2}\right)\right)^2} = a\left(2-\sqrt{2}\right)$$

It is a simple matter to show further that if we now take the bottom right of the square as our origin that the small circle is $2a\left(2-\sqrt{2}\right)$ *up* the diagonal and the large circle is $a\left(2-\sqrt{2}\right)$ *up* the diagonal.